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Petridis, Yiannis N.; Risager, Morten S.

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# LOCAL AVERAGE IN HYPERBOLIC LATTICE POINT COUNTING

YIANNIS N. PETRIDIS AND MORTEN S. RISAGER,  
WITH AN APPENDIX BY NIKO LAAKSONEN

**ABSTRACT.** The hyperbolic lattice point problem asks to estimate the size of the orbit  $\Gamma z$  inside a hyperbolic disk of radius  $\cosh^{-1}(X/2)$  for  $\Gamma$  a discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ . Selberg proved the estimate  $O(X^{2/3})$  for the error term for cofinite or cocompact groups. This has not been improved for any group and any center. In this paper local averaging over the center is investigated for  $\mathrm{PSL}_2(\mathbb{Z})$ . The result is that the error term can be improved to  $O(X^{7/12+\epsilon})$ . The proof uses surprisingly strong input e.g. results on the quantum ergodicity of Maaß cusp forms and estimates on spectral exponential sums. We also prove omega results for this averaging, consistent with the conjectural best error bound  $O(X^{1/2+\epsilon})$ . In the appendix the relevant exponential sum over the spectral parameters is investigated.

## 1. INTRODUCTION

Let  $d$  be the hyperbolic distance on the upper half-plane  $\mathbb{H}$ , and  $u$  the standard point-pair invariant

$$u(z, w) = \frac{|z - w|^2}{4\Im z \Im w}.$$

Let  $\Gamma$  be a discrete subgroup of  $\mathrm{PSL}_2(\mathbb{R})$ , the group of isometries of  $\mathbb{H}$ . Let  $N(z, w, X)$  be defined as

$$(1.1) \quad N(z, w, X) = \#\{\gamma \in \Gamma, 4u(\gamma z, w) + 2 \leq X\}$$

the condition being equivalent to  $d(\gamma z, w) \leq \cosh^{-1}(X/2)$ , i.e.  $N(z, w, X)$  counts the number of lattice points  $\gamma z$  within the hyperbolic circle of radius  $R = \cosh^{-1}(X/2)$  centered at  $w$ . Understanding this function is traditionally called the hyperbolic lattice point problem. This problem has a long history, [5, 12, 22, 8, 23, 16], and several generalizations [1, 18, 7]. We restrict our attention to the case where  $\Gamma$  is cocompact or cofinite. Unlike the Euclidean lattice point problem, there is no known elementary or geometric way of finding asymptotics for the counting function, as the length and the area of a hyperbolic circle are of the same order of growth.

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The problem is related to the pre-trace formula or eigenfunction expansion of an integral kernel for  $\Gamma \backslash \mathbb{H}$ . The main term in the asymptotic expansion is

$$M(z, w, X) = \sqrt{\pi} \sum_{s_j \in (1/2, 1]} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z) \overline{u_j(w)} X^{s_j},$$

where  $\lambda_j = s_j(1-s_j)$  are the small eigenvalues of the automorphic Laplacian, i.e. are less than  $1/4$ . A central problem is to understand the growth of the error term i.e. of  $N(z, w, X) - M(z, w, X)$ . The best known error bound is

$$(1.2) \quad N(z, w, X) = M(z, w, X) + O(X^{2/3}),$$

due to Selberg (1970's unpublished, see [16, Theorem 12.1] and also [6]). This bound holds for *any* cofinite group, and no group with better bound is known. For congruence groups, like e.g.  $\mathrm{PSL}_2(\mathbb{Z})$  the error term is conjectured to be of the order  $O(X^{1/2+\varepsilon})$ . If true, this is essentially optimal possibly up to changing  $X^\varepsilon$  with powers of  $\log \log X$  (See [23]).

For the rest of the paper we restrict to  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ . In this case the main term simplifies to  $M(z, w, X) = \frac{\pi}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} X$ , as there are no small eigenvalues. We investigate *local averages* of the hyperbolic lattice point counting, i.e. we vary the center of the hyperbolic circle locally. We get an improvement on the exponent  $2/3$  on average. To be precise we study the function

$$(1.3) \quad N_f(X) = \int_{\Gamma \backslash \mathbb{H}} f(z) N(z, z, X) d\mu(z),$$

where  $f$  is a smooth, compactly supported function on  $\Gamma \backslash \mathbb{H}$ . We prove the following theorem.

**Theorem 1.1.** *Let  $\Gamma$  be  $\mathrm{PSL}_2(\mathbb{Z})$ , and assume that  $f$  is compactly supported on  $\Gamma \backslash \mathbb{H}$ , smooth and nonnegative. Then*

$$N_f(X) = \pi X \bar{f} + O_{f,\varepsilon}(X^{7/12+\varepsilon}),$$

$$\text{where } \bar{f} = \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu(z).$$

It is tempting to speculate that also in the case of local averaging the order of growth of the error term should be  $X^{1/2+\varepsilon}$ , i.e.

$$(1.4) \quad N_f(X) = \pi X \bar{f} + O_{f,\varepsilon}(X^{1/2+\varepsilon}).$$

We prove an omega result, which is consistent with (1.4):

**Theorem 1.2.** *Let  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ , and assume that  $f$  is a nonzero, nonnegative, smooth, compactly supported function on  $\Gamma \backslash \mathbb{H}$ . Then for every  $\nu > 0$  we have*

$$N_f(X) = \pi X \bar{f} + \Omega(X^{1/2}(\log \log X)^{1/4-\nu}).$$

**Remark 1.3.** Theorem 1.1 improves (on average) the error term (1.2), but only by going *halfway* between Selberg's exponent  $2/3$  and the expected one  $1/2 + \varepsilon$ . Our proof is only valid for groups similar to  $\mathrm{PSL}_2(\mathbb{Z})$ , as it requires surprisingly strong arithmetic input not available for most groups. Among other input we use effective error terms on average for the mass equidistribution of Maaß cusp forms (see Theorem 3.1 below), which itself follows from a remarkable result of Luo–Sarnak on the mean Lindelöf hypothesis for Rankin–Selberg convolutions in the spectral aspect [20].

**Remark 1.4.** Using a spectral large sieve, Chamizo [2, 3] proved that the mean square over the interval  $[X, 2X]$  of the error term is of the expected size, i.e he proved

$$(1.5) \quad \left( \frac{1}{X} \int_X^{2X} |N(z, w, X) - M(z, w, X)|^2 dX \right)^{1/2} = O(X^{1/2} \log X).$$

In fact he proved a more precise statement, namely that if one fixes  $z, w$  and takes sufficiently many, sufficiently well-spaced radii, then the second moment of the absolute value of the error term has the average bound consistent with the optimal error term  $O(X^{1/2+\varepsilon})$  (up to  $X^\varepsilon$  being replaced by powers of  $\log X$ ). The  $L^2$ -estimate (1.5) follows easily from this.

**Remark 1.5.** Theorem 1.2 is a local average analogue of the pointwise omega result proved by Phillips and Rudnick [23, Theorem 1.2]. Our proof follows to a large extent that of [23, Theorem 1.2], with some differences, due to the non-uniformity in  $(z, w)$  of their result. We also make extensive use of known properties of certain special functions simplifying some arguments in their proof. The essential idea of the proof goes back to Hardy [9, p. 23–25].

**Remark 1.6.** There is another approach to the hyperbolic lattice point counting for  $\mathrm{PSL}_2(\mathbb{Z})$ , due to Huxley and Zhigljavsky [14] using Farey fractions. They get an asymptotic formula for the number of pairs of consecutive fractions in the Farey sequence subject to certain restrictions. This approach has not been investigated further in comparison with the application of the spectral theory of automorphic forms.

**Remark 1.7.** Hill and Parnowski in [11] studied the variance of  $N(z, w, X) - \pi X / \mathrm{vol}(\Gamma \backslash \mathbb{H})$  in the  $w$  variable. To simplify their result, we assume that there are no eigenvalues  $\lambda_j \leq 1/4$ ,  $\Gamma$  is cocompact and that we work in the two-dimensional hyperbolic space. Then

$$\int_{\Gamma \backslash \mathbb{H}} \left| N(z, w, X) - \frac{\pi}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} X \right|^2 d\mu(w) = O(X),$$

see [11, Eq. (8)].

**Remark 1.8.** We set  $N(R) = \#\{(m, n) \in \mathbb{Z}^2, m^2 + n^2 \leq R^2\}$  and  $E(R) = N(R) - \pi R^2$ . The function  $N(R)$  is counting the average number  $r(n)$

of representations of an integer  $n$  as sum of two squares. The Gauss circle problem asks to estimate  $E(R)$ . Hardy's conjecture (unproved) states that  $E(R) = O_\varepsilon(R^{1/2+\varepsilon})$ , while the best upper bound known is  $E(R) = O_\varepsilon(R^{131/208+\varepsilon})$ , due to Huxley [13]. Hardy [9] has proved the omega result  $E(R) = \Omega(R^{1/2} \log^{1/4} R)$ . Cramér [4] provided mean-square asymptotics for  $E(R)$ :

$$\int_1^R E(x)^2 dx = c \cdot R^2 + O(R^{3/2}), \quad c = \frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{r(n)^2}{n^{3/2}}.$$

**Remark 1.9.** The structure of the paper is as follows. We discuss background material on exponential sums over the eigenvalues, the rate of quantum ergodicity of eigenfunctions, the pre-trace formula, and approximations to the hyperbolic lattice-point problem in Sections 2, 3, 4, 5. The proof of Theorem 1.1 is in Section 6 and the proof of Theorem 1.2 in Section 7.

## 2. PRIME GEODESIC THEOREMS AND EXPONENTIAL SUMS OVER THE EIGENVALUES

The hyperbolic lattice point problem and the prime geodesic theorem on hyperbolic surfaces are two problems where the spectral theory of automorphic forms can be used, via the pre-trace resp. trace formula. To get good error terms in the prime geodesic theorem Iwaniec [15] proved the following explicit formula for  $\mathrm{PSL}_2(\mathbb{Z})$ :

$$\sum_{N(P) \leq x} \log N(P_0) = x + \sum_{|t_j| \leq T} \frac{x^{s_j}}{s_j} + O\left(\frac{x}{T} \log^2 x\right)$$

for  $T \leq x^{1/2}(\log x)^{-2}$ . Here  $P$  is a conjugacy class of a hyperbolic element,  $P_0$  is the related primitive conjugacy class, and  $N(P_0)$  is its norm. This shows that one cannot expect an error term better than  $x^{3/4+\varepsilon}$  without some cancellation in the sum over eigenvalues, due to Weyl's law ([16, Corollary 11.2]). Let us define

$$(2.1) \quad S(T, X) = \sum_{|t_j| \leq T} X^{it_j}.$$

Using Weyl's law, we have the trivial estimate

$$(2.2) \quad S(T, X) = O(T^2).$$

Iwaniec [15] proved that

$$(2.3) \quad S(T, X) = O(X^{11/48+\varepsilon} T),$$

from which he deduced that

$$\pi(x) = \mathrm{li}(x) + O(x^{35/48+\varepsilon}),$$

where  $\pi(x) = \{P_0, N(P_0) \leq x\}$ . Luo–Sarnak [20, Theorem 1.2] proved the following result.

**Theorem 2.1.** *For the exponential sum (2.1) the following estimate holds:*

$$S(T, X) = O(X^{1/8} T^{5/4} (\log T)^2).$$

We notice that the exponent of  $X$  is smaller than in (2.3) while the exponent of  $T$  is larger. Theorem 2.1 allowed Luo and Sarnak to prove

$$\pi(x) = \text{li}(x) + O(x^{7/10+\varepsilon}).$$

Very recently Soundararajan and Young [27] proved that

$$\pi(x) = \text{li}(x) + O(x^{25/36+\varepsilon})$$

for  $\text{PSL}_2(\mathbb{Z})$  with an entirely different method. One aim of this work is to show how to use cancellation in the exponential sum  $S(T, X)$  in the hyperbolic lattice point problem. We conjecture square root cancellation in (2.1) with uniform dependence on  $X$ :

**Conjecture 2.2.** Let  $X > 2$ . For the exponential sum  $S(T, X)$  we have

$$S(T, X) = O(X^\varepsilon T^{1+\varepsilon}).$$

This conjecture will give the best possible error term in the prime geodesic theorem up to powers of  $\log X$ . We will see that Conjecture 2.2 implies also the best possible error term on average for the hyperbolic lattice point problem, i.e. (1.4). In fact we shall see in Remark 6.4 that

$$(2.4) \quad S(T, X) = O(X^\varepsilon T^{3/2-\delta})$$

for some  $\delta > 0$  suffices to prove (1.4).

In the Appendix, N. Laaksonen investigates the conjecture numerically and proves a theorem for the exponential sum  $S(T, X)$  for fixed  $X$ , as  $T \rightarrow \infty$ . The numerics and the theorem point to the correctness of Conjecture 2.2.

### 3. QUANTUM ERGODICITY

For  $u_j$  an orthonormal basis of eigenfunctions we form the measures

$$(3.1) \quad d\mu_j = |u_j(z)|^2 d\mu(z).$$

For  $\text{PSL}_2(\mathbb{Z})$  Lindenstrauss [19] and Soundararajan [26] have proved recently that for Hecke-Maaß eigenforms the Quantum Unique Ergodicity conjecture holds, i.e.

$$(3.2) \quad d\mu_j \rightarrow \frac{1}{\text{vol}(\Gamma \backslash \mathbb{H})} d\mu(z), \quad j \rightarrow \infty.$$

The question of the *rate* of convergence of (3.2) has been raised by Sarnak [24, Eq. 3.7], who conjectured that

$$(3.3) \quad \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} = O(t_j^{-1/2+\varepsilon}),$$

where  $\bar{f} = \text{vol}(\Gamma \backslash \mathbb{H})^{-1} \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu(z)$ . For general hyperbolic surfaces Zelditch [29] proved

$$\sum_{|t_j| \leq T} \left| \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right|^{2k} = O\left(\frac{T^2}{\log^k T}\right).$$

For  $\text{PSL}_2(\mathbb{Z})$  Luo–Sarnak [20] proved the optimal bound (3.3) on average:

**Theorem 3.1.** *Let  $f$  be a smooth compactly supported function on  $\Gamma \backslash \mathbb{H}$ .*

$$\sum_{|t_j| \leq T} \left| \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right|^2 = O(\|f\|_{8,8}^2 T^{1+\varepsilon}),$$

where the constant depends only on  $\varepsilon$ .

The norm  $\|f\|_{8,8}$  is finite for  $f$  smooth and compactly supported. We use Theorem 3.1 in Lemma 6.2 to get rid of the eigenfunctions in the integrated pre-trace formula (see Proposition 4.1 below).

#### 4. INTEGRATED PRE-TRACE FORMULA

Let  $k \in \mathbb{C}^\infty(\mathbb{R}_+)$  be a function with Selberg–Harish-Chandra transform  $h(t)$  (see [16, (1.62)] for its definition) even, holomorphic in  $|\Im t| \leq 1/2 + \varepsilon$ , and  $h(t) = O(1/(1+|t|)^{2+\varepsilon})$  in the strip.

Let

$$(4.1) \quad K(z, w) = \sum_{\gamma \in \Gamma} k(u(\gamma z, w))$$

be the corresponding automorphic kernel, and  $K$  the corresponding integral operator.

**Proposition 4.1.** *Let  $f$  be a smooth, compactly supported function on  $\Gamma \backslash \mathbb{H}$ . Then we have*

$$\begin{aligned} \int_{\Gamma \backslash \mathbb{H}} f(z) K(z, z) d\mu(z) &= \bar{f} \sum_{t_j} h(t_j) + \sum_{t_j} h(t_j) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right) \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}} h(t) \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) dt \end{aligned}$$

with absolute convergence on the right-hand side.

*Proof.* We set  $z = w$  in Selberg’s pre-trace formula [16, Theorem 7.4], [25] and integrate against  $f$ .  $\square$

**Remark 4.2.** We note that the integral  $\int_{\Gamma \backslash \mathbb{H}} f(z) K(z, z) d\mu(z)$  can be interpreted as the trace of the operator  $M_f K$ , where  $M_f$  is multiplication by  $f$ . The operator  $M_f K$  has kernel  $f(z) K(z, w)$ , which is *not* a point-pair invariant for  $\text{PSL}_2(\mathbb{R})$ .

**Remark 4.3.** We remark that the first term on the right-hand side of Proposition 4.1 is  $\bar{f}$  times the contribution of the discrete spectrum to the Selberg trace formula [16, Theorem 10.2], [25]. We shall see that for groups like  $\mathrm{PSL}_2(\mathbb{Z})$  the two last terms can be estimated using Theorem 3.1 and the Maaß–Selberg relations.

## 5. APPROXIMATION IN THE HYPERBOLIC LATTICE POINT PROBLEM

Let  $\chi_A$  denote the characteristic function of a set  $A$ . One would like to use  $k(u) = \chi_{[0, (X-2)/4]}(u)$  in the pre-trace formula, since the corresponding integral kernel is exactly  $N(z, w, X)$ . Unfortunately the decay in  $t$  of the corresponding Selberg–Harish-Chandra transform  $h(t)$  is not strong enough to analyze effectively the right-hand side of the pre-trace formula. Therefore, it is better to smooth  $k(u) = \chi_{[0, (X-2)/4]}(u)$ . Various types of smoothing are appropriate depending on the problem at hand.

For kernels  $k_1$  and  $k_2$  their *hyperbolic convolution* [3] is defined as

$$k_1 * k_2(u(z, w)) = \int_{\mathbb{H}} k_1(u(z, v)) k_2(u(v, w)) d\mu(v).$$

The Selberg–Harish-Chandra transform of the convolution is the pointwise product of the individual Selberg–Harish-Chandra transforms, i.e.

$$h_{k_1 * k_2}(t) = h_{k_1}(t) \cdot h_{k_2}(t),$$

see [3, p. 323]. In this paper we will use the (non-smooth) mollifier

$$k_\delta(u) = \frac{1}{4\pi \sinh^2(\delta/2)} \chi_{[0, (\cosh \delta - 1)/2]}(u).$$

with ‘small’ parameter  $\delta$ . This kernel satisfies  $\int_{\mathbb{H}} k_\delta(u(z, w)) d\mu(z) = 1$ . The main reason for using this mollifier rather than a smooth one is that we can compute its Selberg–Harish-Chandra transform explicitly. Indeed for any indicator function  $\chi_{[0, (\cosh R - 1)/2]}(u)$  its transform equals

$$(5.1) \quad h(t) = 2^{5/2} \int_0^R (\cosh R - \cosh r)^{1/2} \cos(rt) dr = 2\pi \sinh(R) P_{-1/2+it}^{-1}(\cosh R),$$

where  $P_\mu^\nu(z)$  is the associated Legendre function of the first kind. Many properties of the kernels we shall choose later follow from (5.1). Lemma 2.4 in [3] gives the following estimates:

$$(5.2) \quad h(t) = O((R+1)e^{R/2})$$

uniformly for  $t$  real. Furthermore

$$h(t) = 2|t|^{-3/2} \sqrt{2\pi \sinh R} \cos(Rt - (3\pi/4)\mathrm{sgn} t) + O(t^{-5/2} e^{R/2})$$

for  $t$  real,  $|t| \geq 1$ , and  $R \geq 1$ . For  $0 \leq R \leq 1$  and  $t$  real and  $|t| \geq 1$  we have

$$h(t) = 2\pi R t^{-1} J_1(Rt) \sqrt{\frac{\sinh R}{R}} + O(R^2 \min(R^2, |t|^{-2})),$$



where  $J_1$  is the Bessel function of order 1. Moreover, for every  $R > 0$  we have  $h(i/2) = 2\pi(\cosh R - 1)$ . The value  $t = i/2$  corresponds to the eigenvalue  $\lambda = 0$ , which gives the main term for  $\mathrm{PSL}_2(\mathbb{Z})$ , as  $\mathrm{PSL}_2(\mathbb{Z})$  has no small eigenvalues. In general we can use [3, Lemma 2.4 (b)] to analyze the contribution of small eigenvalues.

Given  $X > 0$  we define  $R$  to be the positive solution of the equation  $1 + 2X = \cosh R$ . We also define  $Y$  through  $\cosh Y = X/2$  with  $Y > 0$ . With these definitions  $u \leq (X - 2)/4$  precisely when  $d \leq Y$ .

Given  $Z > 0$ , using the triangle inequality for the hyperbolic distance,  $d(z, w) \leq d(z, v) + d(v, w)$ , it is straightforward to verify that

$$(5.3) \quad \chi_{[0, (\cosh(Z)-1)/2]} * k_\delta(u(z, w)) = \begin{cases} 1, & \text{if } d(z, w) \leq Z - \delta, \\ 0, & \text{if } d(z, w) \geq Z + \delta. \end{cases}$$

We now define functions with values in  $[0, 1]$  by

$$k_\pm := \chi_{[0, (\cosh(Y \pm \delta)-1)/2]} * k_\delta$$

and denote the corresponding Selberg–Harish-Chandra transforms by  $h_\pm$ . Using (5.3) we now see that

$$(5.4) \quad k_-(u) \leq \chi_{[0, (X-2)/4]}(u) \leq k_+(u).$$

This inequality allows to pass from smoothed kernels to the sharp cut-off. In the following  $X$  will be a large parameter tending to infinity, and  $\delta > 0$  a small parameter tending to zero, given as a function of  $X$ . We notice that  $\sinh(Y \pm \delta) = O(X)$ . Using the above general bounds for  $h(t)$  we have for  $0 < \delta < 1$

$$h_\delta(t) = \frac{1}{2} \frac{\sqrt{\delta \sinh(\delta)}}{|t| \sinh^2(\delta/2)} J_1(\delta |t|) + O(\delta^2 \min(1, (\delta |t|)^{-2})),$$

where  $h_\delta(t) := h_{k_\delta}(t)$ . Therefore, we have the following estimates for the Selberg–Harish-Chandra transforms  $h_\pm$  of  $k_\pm$  that are valid for  $t$  real and  $|t| \geq 1$ , and  $Y - \delta > 1$ :

$$(5.5) \quad h_\pm(t) = H_\pm(t) + O\left(\frac{X^{1/2}}{|t|^{3/2}} \left(\delta^2 \min(1, (\delta |t|)^{-2}) + |t|^{-1} \min(1, (\delta |t|)^{-3/2})\right)\right),$$

where

$$(5.6) \quad H_\pm(t) = \frac{\sqrt{2\pi\delta \sinh(\delta) \sinh(Y \pm \delta)}}{|t|^{5/2} \sinh^2(\delta/2)} J_1(\delta |t|) \cos((Y \pm \delta)t - (3\pi/4)\mathrm{sgn}t).$$

The error term is found by multiplying the error term for  $h_\delta$  by a bound for the main term of the transform  $h$  of  $\chi_{[0, (\cosh(Y \pm \delta)-1)/2]}$  and then adding a bound for the main term for  $h_\delta$  with the error term for  $h$ .

Using Weyl's law and the above bounds, it is straightforward to verify that for  $\delta$  bounded we have

$$(5.7) \quad \sum_{|t_j| \geq 1} (h_{\pm}(t_j) - H_{\pm}(t_j)) = O(X^{1/2}).$$

We notice that by writing the cosine in  $H_{\pm}(t)$  as a sum of exponentials we can write

$$(5.8) \quad H_{\pm}(t) = A_{\pm}(t, X, \delta) e^{it(Y \pm \delta)} + B_{\pm}(t, X, \delta) e^{-it(Y \pm \delta)}.$$

We use

$$(5.9) \quad J_1(z) = O(\min(|z|, |z|^{-1/2})), \quad J'_1(z) = O(\min(1, |z|^{-1/2})),$$

see [16, Appendix B4] to get for  $|t| \geq 1$

$$(5.10) \quad A_{\pm}(t, X, \delta), B_{\pm}(t, X, \delta) = O(X^{1/2} |t|^{-3/2} \min(1, (\delta |t|)^{-3/2}))$$

and

$$(5.11) \quad A'_{\pm}(t, X, \delta), B'_{\pm}(t, X, \delta) = O(X^{1/2} |t|^{-5/2} \min(1, (\delta |t|)^{-1/2})).$$

**Remark 5.1.** The smoothed functions  $h_{\pm}(t)$  decay as  $O(|t|^{-5/2})$ , which is better than the rate of decay of the non-smooth ones i.e.  $O(|t|^{-3/2})$ . If we use  $k_{\delta} * \dots * k_{\delta}$  rather than just  $k_{\delta}$ , the rate of decay becomes even better (an additional  $|t|^{-1}$  for each extra convolution). Unfortunately this does not improve the final bound. Notice that  $1 * k_{\delta} * \dots * k_{\delta} = 1$ , i.e.  $k_{\delta} * \dots * k_{\delta}$  has integral 1.

## 6. UPPER BOUNDS

In this section we will prove Theorem 1.1. To do so we will use the kernels  $k_{\pm}$  constructed in the previous section. We will assume that  $\delta = X^{-c}$  for some  $c > 0$ .

We analyzing the various terms of the spectral side in Proposition 4.1.

We start evaluating the contribution of the continuous spectrum to the integrated pre-trace formula.

**Lemma 6.1.** *If  $\Gamma$  is a congruence group*

$$\int_{\mathbb{R}} h_{\pm}(t) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) \right) dt = O(X^{1/2} \log X).$$

*Proof.* By using that  $f$  has support in

$$F_a = \{z \in \mathbb{H}, |z| > 1, |\Re(z)| \leq 1/2, \Im(z) \leq a\}$$

for  $a$  sufficiently large, we see that

$$\begin{aligned} \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) &\leq C \int_{F_a} |E(z, 1/2 + it)|^2 d\mu(z) \\ &\leq C \|E^a(z, 1/2 + it)\|^2, \end{aligned}$$

where  $E^a(z, 1/2 + it)$  is the truncated Eisenstein series (see [25, (7.39)]). By the Maaß–Selberg relations we have

$$\|E^a(z, 1/2 + it)\|^2 = O_a \left( 1 + \left| \frac{-\phi'}{\phi}(1/2 + it) \right| \right),$$

see [25, (7.42')]. For congruence groups the scattering matrix can be computed, and this leads to

$$(6.1) \quad \frac{-\phi'}{\phi} \left( \frac{1}{2} + it \right) = O(\log(2 + |t|)),$$

see e.g. [10, Eq. 2.5 p.508] for  $\mathrm{PSL}_2(\mathbb{Z})$ . It follows that

$$(6.2) \quad \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) = O_f(\log(2 + |t|)).$$

Using (5.2), (5.5), and (5.10), we get

$$(6.3) \quad h_{\pm}(t) = O(X^{1/2} |t|^{-3/2})$$

for  $|t| \geq 1$ . It follows that

$$\int_{|t| > 1} h_{\pm}(t) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) \right) dt = O(X^{1/2}).$$

For  $|t| < 1$  we use  $h_{\pm}(t) = O(X^{1/2} \log X)$ , see (5.2). We deduce that

$$\int_{|t| \leq 1} h_{\pm}(t) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) \right) dt = O(X^{1/2} \log X).$$

□

**Lemma 6.2.**

$$\sum_{t_j} h_{\pm}(t_j) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right) = O(X^{1/2+\varepsilon}).$$

*Proof.* The eigenvalue  $\lambda = 0$  does not contribute since

$$\int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_0(z) - \bar{f} = 0.$$

Therefore, using (5.2), we only need to sum over  $|t_j| \geq 1$ . This can be bounded by using Theorem 3.1, the Cauchy–Schwarz inequality, Weyl’s law,

and dyadic decomposition. We find

$$\begin{aligned} & \left| \sum_{T < |t_j| \leq 2T} h_{\pm}(t_j) \left( \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right) \right| \\ & \leq \left( \sum_{T < |t_j| \leq 2T} |h_{\pm}(t_j)|^2 \right)^{1/2} \left( \sum_{T < |t_j| \leq 2T} \left| \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) - \bar{f} \right|^2 \right)^{1/2} \\ & = O_f \left( \max_{T \leq t \leq 2T} |h_{\pm}(t)| T^{3/2+\varepsilon} \right). \end{aligned}$$

Using

$$H_{\pm}(t) = O(|t|^{-3/2} X^{1/2} \min(1, (\delta |t|)^{-3/2})),$$

which follows from (5.10), and (5.5) we find

$$(6.4) \quad O_f \left( \max_{T \leq t \leq 2T} |h_{\pm}(t)| T^{3/2+\varepsilon} \right) = O_f(T^{\varepsilon} X^{1/2} \min(1, (\delta T)^{-3/2})).$$

We now observe that the dyadic sums over  $T = 2^n$  satisfies

$$\sum_{n=0}^{\infty} 2^{n\varepsilon} X^{1/2} \min(1, (\delta 2^n)^{-3/2}) = O(X^{1/2+\varepsilon}).$$

This can be deduced by splitting the sum at  $\delta 2^n = 1$  and computing the resulting geometric series, and using  $\delta = X^{-c}$  for some  $c > 0$ . The result follows.  $\square$

**Lemma 6.3.** *For  $\Gamma = PSL_2(\mathbb{Z})$  the following estimate holds:*

$$(6.5) \quad \sum_{t_j \in \mathbb{R}} h_{\pm}(t_j) + \frac{1}{4\pi} \int_{\mathbb{R}} h_{\pm}(t) \frac{-\phi'}{\phi} \left( \frac{1}{2} + it \right) dt = O(X^{7/12+\varepsilon}).$$

*Proof.* Using (6.1), (5.2), (6.3) we prove that the integral is  $O(X^{1/2+\varepsilon})$ . This is similar to the last part of the proof of Lemma 6.1. There are finitely many terms with  $|t_j| < 2$  and each is  $O(X^{1/2} \log X)$  by (5.2). So we need to estimate the sum

$$\sum_{2 \leq |t_j|} h_{\pm}(t_j).$$

By (5.7) we see that it suffices to bound

$$\sum_{2 \leq |t_j|} H_{\pm}(t_j).$$

We now consider

$$(6.6) \quad \sum_{T < |t_j| \leq 2T} H_{\pm}(t_j).$$

Using (5.8) this equals

$$\sum_{T < |t_j| \leq 2T} A_{\pm}(t_j, X, \delta) e^{it_j(Y \pm \delta)}$$

plus a similar expression with  $B$  instead of  $A$ . We recall the definition (2.1) and use summation by parts to write the sum as

$$A_{\pm}(2T, X, \delta)S(2T, e^{Y \pm \delta}) - A_{\pm}(T, X, \delta)S(T, e^{Y \pm \delta}) - \int_T^{2T} A'_{\pm}(u, X, \delta)S(u, e^{Y \pm \delta})du.$$

Therefore, using (5.10), (5.11),  $e^{Y \pm \delta} = O(X)$ , and *any* bound of the form

$$S(T, X) = O(X^a T^b),$$

we find that (6.6) is estimated as

$$O(X^{1/2+a} T^{b-3/2} \min(1, (\delta T)^{-1/2})).$$

By summing over dyadic intervals we get

$$\begin{aligned} \sum_{2 \leq |t_j|} H_{\pm}(t_j) &= O \left( X^{1/2+a} \sum_{n=1}^{\infty} 2^{n(b-3/2)} \min(1, (\delta 2^n)^{-1/2}) \right) \\ (6.7) \quad &= O \left( X^{1/2+a} \left( \sum_{n \leq -\log_2(\delta)} 2^{n(b-3/2)} + \sum_{n > -\log_2(\delta)} 2^{n(b-2)} \delta^{-1/2} \right) \right). \end{aligned}$$

The first sum is bounded as long as  $b < 3/2$ , while second sum is  $O(\delta^{3/2-b})$  if  $b < 2$ . We therefore find that as long as  $b < 3/2$  then (6.7) is  $O(X^{1/2+a})$ .

To get a good bound for  $S(T, X)$  we interpolate between the trivial bound (2.2) and the Luo–Sarnak bound (Theorem 2.1). To optimize we use the elementary inequality

$$(6.8) \quad \min(k, l) \leq k^r l^{1-r}$$

valid for  $k, l > 0$ ,  $0 \leq r \leq 1$ .

We find that for any  $0 \leq r \leq 1$

$$S(T, X) = O((X^{\frac{r}{8}} T^{r\frac{5}{4}+2(1-r)+\varepsilon}).$$

We let  $a(r) = \frac{r}{8}$  and  $b(r) = r\frac{5}{4} + 2(1-r)$ . For every  $r > 2/3$  we have  $b(r) < 3/2$ . It follows that for every  $r > 2/3$  the sum in (6.7) is  $O_r(X^{1/2+r/8+\varepsilon})$ . The result now follows.  $\square$

**Remark 6.4.** We notice that if we knew (2.4) then the above proof would give the optimal exponent  $1/2 + \varepsilon$  instead of  $7/12 + \varepsilon$ . We note also that if we balance the error term from the zero eigenvalue, i.e.  $O(X\delta)$  with the second error term  $O(X^{1/2+a}\delta^{3/2-b})$  in (6.7) with  $(a, b) = (1/8, 5/4 + \varepsilon)$  from Theorem 2.1 we find  $\delta = X^{-1/2+\varepsilon}$ . This optimizes the error terms.

We can now prove Theorem 1.1: We observe that

$$(6.9) \quad h_{\pm}(i/2) = 2\pi(\cosh(Y \pm \delta) - 1) \frac{2\pi(\cosh(\delta) - 1)}{4\pi \sinh^2(\delta/2)} = \pi X + O(X\delta).$$

Using Lemmata 6.1, 6.2, and 6.3, and Proposition 4.1 we find that

$$\int_{\Gamma \backslash \mathbb{H}} f(z) K_{\pm}(z, z) = \bar{f} \pi X + O(X^{7/12+\varepsilon}) + O(X\delta).$$

This gives by (5.4)

$$\int_{\Gamma \backslash \mathbb{H}} f(z) \sum_{\gamma \in \Gamma} 1_{[0, (X-2)/4]}(u(\gamma z, z)) d\mu(z) = \bar{f} \pi X + O(X^{7/12+\varepsilon}) + O(X\delta),$$

since  $f$  is positive. Theorem 1.1 follows by choosing  $\delta = X^{-1/2}$ .

## 7. OMEGA RESULTS

In this section we investigate omega results for  $N_f(X)$ .

**Theorem 7.1.** *Let  $f$  be a nonzero, nonnegative, compactly supported function on  $\Gamma \backslash \mathbb{H}$ . Then for every  $\nu > 0$  we have*

$$N_f(X) = \pi X \bar{f} + O(X^{1/2}(\log \log x)^{1/4-\nu}).$$

Since analogous omega results hold pointwise (see [23, Theorem 1.2]) this is not a surprising result. In fact our proof below is based on investigating the uniformity in  $z$  in the proof in [23, Theorem 1.2]. The main ingredients in [23, Theorem 1.2] are two lemmata. The first assures that certain phases can be aligned, and the second provides asymptotics (and in particular omega results) for an ‘average local Weyl law’. We quote the alignment lemma directly from [23, Lemma 3.3]. It can be proved using a simple application of Dirichlet’s box principle and the elementary inequality  $|e^{i\theta} - 1| < |\theta|$  for  $\theta \neq 0$ :

**Lemma 7.2.** *Given  $n$  real numbers  $r_1, \dots, r_n$ ,  $M > 0$ , and  $T > 1$ , there exists an  $s$  with  $M \leq s \leq MT^n$  such that*

$$|e^{ir_j s} - 1| < \frac{1}{T}, \quad j = 1, \dots, n.$$

The ‘average local Weyl law’ is slightly more subtle, since the main term in the ‘local Weyl law’ proved in [23, Lemma 2.3] depends on the point  $z$ :

**Lemma 7.3.** *Let  $f$  be a smooth compactly supported function on  $\Gamma \backslash \mathbb{H}$ . Then*

$$\sum_{|t_j| \leq T} \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^T \int_{\Gamma \backslash \mathbb{H}} f(z) |E_{\mathfrak{a}}(z, 1/2 + it)|^2 d\mu(z) dt \sim \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} \bar{f} T^2$$

as  $T \rightarrow \infty$ .

*Sketch of proof.* This is a more or less standard application of the heat kernel and a Tauberian theorem. For  $\delta > 0$  we let  $h(t) = e^{-\delta t^2}$ . Using this as the spectral kernel in the pre-trace formula and integrating on the diagonal against  $f$  we obtain

$$\begin{aligned} & \sum_{\gamma \in \Gamma} \int_{\Gamma \backslash \mathbb{H}} f(z) k(u(\gamma z, z)) d\mu(z) \\ (7.1) \quad &= \sum_{t_j} h(t_j) \int_{\Gamma \backslash \mathbb{H}} f(z) d\mu_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} h(t) \int_{\Gamma \backslash \mathbb{H}} f(z) |E_{\mathfrak{a}}(z, 1/2 + it)|^2 d\mu(z) dt, \end{aligned}$$

where  $k$  is the inverse Selberg–Harish-Chandra transform of  $h$ . The function  $k(u)$  is decreasing in  $u$  and satisfies

$$(7.2) \quad k\left(\frac{\cosh v - 1}{2}\right) \leq \frac{C}{\delta} e^{-\frac{v^2}{4\delta}},$$

where  $C$  is an absolute constant. This follows from an elementary evaluation of  $k$  (See [2, Lemma 3.1]). Furthermore

$$(7.3) \quad k(0) = \frac{1}{4\pi} \int_{\mathbb{R}} t \tanh(\pi t) h(t) dt = \frac{1}{4\pi\delta} + O(1)$$

as  $\delta \rightarrow 0$ , which follows from  $\tanh(\pi t) = 1 + O(e^{-2\pi|t|})$ . We now notice, since  $u(\gamma z, z) = (\cosh(d(\gamma z, z)) - 1)/2$ , that by (7.2)

$$(7.4) \quad \sum_{I \neq \gamma \in \Gamma} \int_{\Gamma \backslash \mathbb{H}} f(z) k(u(\gamma z, z)) d\mu(z) = O\left(\frac{1}{\delta} \int_{K \cap F} \sum_{I \neq \gamma \in \Gamma} e^{-\frac{d(\gamma z, z)^2}{4\delta}} d\mu(z)\right),$$

where  $K$  is the support of  $f$  and  $F$  is some Dirichlet fundamental domain.

We will show that the right-hand side in (7.4) is  $o(1/\delta)$ . This implies that the left-hand side of (7.1) is asymptotic to  $\text{vol}(\Gamma \backslash \mathbb{H}) \bar{f} / (4\pi\delta)$  as  $\delta \rightarrow 0$ . The claim of the theorem now follows from Karamata's Tauberian theorem (see [28, Theorem 4.3]).

To analyze

$$(7.5) \quad \frac{1}{\delta} \int_{K \cap F} \sum_{I \neq \gamma \in \Gamma} e^{-\frac{d(\gamma z, z)^2}{4\delta}} d\mu(z)$$

we split the sum as

$$(7.6) \quad \sum_{I \neq \gamma} e^{-\frac{d(\gamma z, z)^2}{4\delta}} = \sum_{I \neq \gamma \in A} e^{-\frac{d(\gamma z, z)^2}{4\delta}} + \sum_{I \neq \gamma \in \Gamma \backslash A} e^{-\frac{d(\gamma z, z)^2}{4\delta}},$$

where  $A = \{\gamma \in \Gamma \mid d(\gamma w, w) \leq 1 \text{ for some } w \in K \cap F\}$ .

We consider the first sum. We claim that  $A$  is finite. To see this let

$$M = \{z \in \mathbb{H}, \exists w \in K \cap F \text{ with } d(z, w) \leq 1\}.$$

The set  $M$  is compact. Now note that  $B = \{\gamma \in \Gamma, M \cap \gamma F \neq \emptyset\}$  contains  $A$  and is finite by [21, Theorem 1.6.2 (3)]. For  $\gamma_0 \in A$  let  $\epsilon > 0$  and split  $K \cap F$  as

$$K \cap F = F_1(\epsilon) \cup F_2(\epsilon) = \{z \in K \cap F, d(\gamma_0 z, z) \leq \epsilon\} \cup \{z \in K \cap F, d(\gamma_0 z, z) > \epsilon\}.$$

It is now clear that

$$\begin{aligned} \int_{K \cap F} e^{-\frac{d(\gamma_0 z, z)^2}{4\delta}} d\mu(z) &= \int_{F_1(\epsilon)} e^{-\frac{d(\gamma_0 z, z)^2}{4\delta}} d\mu(z) + \int_{F_2(\epsilon)} e^{-\frac{d(\gamma_0 z, z)^2}{4\delta}} d\mu(z) \\ &= O(\mu(F_1(\epsilon))) + O(e^{-\frac{\epsilon^2}{4\delta}}). \end{aligned}$$

By Lemma 7.4 we have  $\mu(F_1(\epsilon)) = O(\epsilon^2)$ . Choosing  $\epsilon = \sqrt{4\delta}(-\log \delta)$  we see that

$$\frac{1}{\delta} \int_{K \cap F} e^{-\frac{d(\gamma_0 z, z)^2}{4\delta}} d\mu(z) = o(1/\delta).$$

Since there are only finitely many terms in the sum over  $A$ , this estimate suffices in dealing with this sum.

To handle the second sum in (7.6) we use that

$$\#\{\gamma \in \Gamma, d(\gamma z, z) \leq r\} = O(e^r),$$

where the implied constant is absolute for  $z$  in a compact set and fixed group  $\Gamma$  (see [16, Corollary 2.12]).

It follows that for  $z \in K$

$$\begin{aligned} \sum_{I \neq \gamma \in \Gamma \setminus A} e^{-\frac{d(\gamma z, z)^2}{4\delta}} &= \sum_{n=0}^{\infty} \sum_{\substack{\gamma \in \Gamma \\ 2^n \leq d(\gamma z, z) \leq 2^{n+1}}} e^{-\frac{d(\gamma z, z)^2}{4\delta}} \\ &\leq \sum_{n=0}^{\infty} e^{-\frac{4^n}{4\delta}} O(e^{2^{n+1}}) = O(e^{-\frac{C}{\delta}}) \end{aligned}$$

for some absolute constant  $C > 0$ . This suffices to conclude that

$$\frac{1}{\delta} \int_{K \cap F} \sum_{I \neq \gamma \in \Gamma \setminus A} e^{-\frac{d(\gamma z, z)^2}{4\delta}} d\mu(z) = o(1/\delta).$$

Collecting all the terms we find that the left-hand side in (7.4) is  $o(\delta^{-1})$ . This concludes the proof.  $\square$

We have not been able to find a reference for the following elementary result:

**Lemma 7.4.** *Let  $M \subseteq \mathbb{H}$  be any set. Let  $\pm I \neq \gamma \in SL_2(\mathbb{R})$ . Then for sufficiently small  $\epsilon$  we have*

$$\mu(\{z \in M, d(\gamma z, z) < \epsilon\}) = \begin{cases} 0, & \text{if } \gamma \text{ is hyperbolic,} \\ 0, & \text{if } \gamma \text{ is parabolic and } M \text{ compact,} \\ O(\epsilon^2), & \text{if } \gamma \text{ is elliptic.} \end{cases}$$

*Proof.* Since  $d(z, w)$  is a point-pair invariant we can assume that  $\gamma$  is in canonical form. We can also assume  $\epsilon < 1$ . If  $d(\gamma z, z) < \epsilon$  then  $u(\gamma z, z) = (\cosh d(\gamma z, z) - 1)/2 \leq \epsilon^2$ .

Let  $\gamma$  be hyperbolic. Then  $\gamma z = pz$  for some real number  $p > 1$ . Then

$$u(\gamma z, z) = \frac{|p-1|^2 |z|^2}{4py^2} \geq \frac{|p-1|^2}{4p},$$

which shows that  $\{z \in M, d(\gamma z, z) < \epsilon\}$  is empty for  $\epsilon < |p-1|/(2\sqrt{p})$ .

Let  $\gamma$  be parabolic and  $M$  compact. Then  $\gamma z = z + v$  for some  $v \in \mathbb{R} \setminus \{0\}$ , so that

$$u(\gamma z, z) = \frac{|v|^2}{4y^2} \geq C$$



for some positive  $C$ . This shows that  $\{z \in M, d(\gamma z, z) < \epsilon\}$  is empty for  $\epsilon < \sqrt{C}$ .

Last let  $\gamma$  be elliptic, i.e.  $\gamma z = \frac{(\cos \theta)z + \sin \theta}{-(\sin \theta)z + \cos \theta}$  for some  $\theta \notin \pi\mathbb{Z}$ . Then

$$(7.7) \quad u(\gamma z, z) = \sin^2 \theta \frac{|1 + z^2|^2}{4y^2}.$$

We claim that for some suitable constant  $C > 0$  depending only on  $\gamma$  we have

$$(7.8) \quad \{z, d(\gamma z, z) < \epsilon\} \subset \{z, |\Re(z)| < C\epsilon \text{ and } |\Im(z) - 1| < C\epsilon\}.$$

As the last set has hyperbolic area  $O(\epsilon^2)$ , this completes the proof of the lemma. To prove the claim (7.8) we note that, if  $d(\gamma z, z) < \epsilon$ , then (7.7) gives

$$\frac{(1 - y^2 + x^2)^2}{y^2} \leq \frac{4\epsilon^2}{\sin^2 \theta} \quad \text{and} \quad \frac{4x^2 y^2}{y^2} \leq \frac{4\epsilon^2}{\sin^2 \theta}.$$

The second inequality shows that  $|\Re(z)| < C\epsilon$ . By the first inequality we have

$$-\frac{2\epsilon}{|\sin \theta|} \leq \frac{(1 - y^2 + x^2)}{y} \leq \frac{2\epsilon}{|\sin \theta|}.$$

The upper inequality shows that  $y$  is bounded away from zero, and the lower inequality shows that  $y$  is bounded. It follows that for some positive constant  $C'$ ,  $-C'\epsilon \leq (1 - y)(1 + y) \leq C'\epsilon$  and in turn  $|1 - y| \leq C''\epsilon$ . This completes the proof of the claim (7.8).  $\square$

**Remark 7.5.** We note that the result by Phillips–Rudnick [23, Lemma 2.3] analogous to Lemma 7.3 is *not* uniform in  $z$ , as the main term depends on the size of the stabilizer of  $z$ . The proof above shows that, when we integrate, the contribution of elliptic points is small.

We now investigate the omega result for  $N_f(X) - \pi \bar{f} X$ . Recall that  $X/2 = \cosh(Y)$ . As we expect the order to be close to  $\sqrt{X} \sim \sqrt{2 \sinh Y}$  we consider

$$(7.9) \quad E(Y) = \frac{N_f(2 \cosh(Y)) - 2\pi \bar{f} \cosh(Y)}{\sqrt{2\pi \sinh Y}}.$$

Similarly to the proof of the upper bound it is convenient to smooth out (7.9). We use a smoothing technique similar to Phillips–Rudnick.

Consider a smooth, even function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$ , satisfying that  $\psi, \hat{\psi} \geq 0$ ,  $\int_{\mathbb{R}} \psi(x) dx = 1$ ,  $\text{supp}(\psi) \subseteq [-1, 1]$ . For  $0 < \epsilon < 1$  we let  $\psi_\epsilon(r) = \epsilon^{-1} \psi(\epsilon^{-1} r)$  which approximates a  $\delta$ -distribution at 0 as  $\epsilon \rightarrow 0$ . We consider the smoothed-out function

$$(7.10) \quad E_\epsilon(Y) = \int_{\mathbb{R}} \psi_\epsilon(R - Y) E(R) dR.$$

By inserting the definition of  $E(R)$  and interchanging sums we find that

$$E_\epsilon(Y) = \int_{\Gamma \backslash \mathbb{H}} f(z) \left( \sum_{\gamma \in \Gamma} \int_{\mathbb{R}} \psi_\epsilon(R - Y) \frac{1}{\sqrt{2\pi \sinh R}} \chi_{[0, (\cosh R - 1)/2]}(u(\gamma z, z)) dR \right. \\ \left. - \int_{\mathbb{R}} \psi_\epsilon(R - Y) \frac{2\pi \cosh(R)}{\sqrt{2\pi \sinh(R)} \text{vol}(\Gamma \backslash \mathbb{H})} dR \right) d\mu(z).$$

We see that the infinite sum over  $\Gamma$  is an automorphic kernel evaluated at the diagonal. The corresponding free kernel is

$$k_{\epsilon, Y}(u) = \int_{\mathbb{R}} \psi_\epsilon(R - Y) \frac{1}{\sqrt{2\pi \sinh R}} \chi_{[0, (\cosh R - 1)/2]}(u) dR,$$

and its corresponding Selberg–Harish-Chandra transform (compare [16, (1.62')]) is

(7.11)

$$h_{\epsilon, Y}(t) = 4\pi \int_0^\infty F_s(u) k_{\epsilon, Y}(u) du = \int_{\mathbb{R}} \psi_\epsilon(R - Y) \frac{1}{\sqrt{2\pi \sinh R}} h_R(t) dR.$$

Here  $s = 1/2 + it$ ,  $F_s(u)$  is the Gauss hypergeometric function  $F(s, 1-s; 1, u)$ , and  $h_R$  is the Selberg–Harish-Chandra transform of  $\chi_{[0, (\cosh R - 1)/2]}(u)$ . We note that the smoothed-out kernel has as Selberg–Harish-Chandra transform the smoothing of the initial transform.

We notice also that

$$h_{\epsilon, Y}(i/2) = \int_{\mathbb{R}} \psi_\epsilon(R - Y) \frac{2\pi(\cosh R - 1)}{\sqrt{2\pi \sinh R}} dR.$$

Assuming that  $h_{\epsilon, Y}(t)$  decays sufficiently fast, which we will verify below, it follows from the pre-trace formula that

(7.12)

$$E_\epsilon(Y) = \int_{\Gamma \backslash \mathbb{H}} f(z) \left( \sum_{t_j \neq i/2} h_{\epsilon, Y}(t_j) |u_j(z)|^2 \right. \\ \left. + \frac{1}{4\pi} \int_{\mathbb{R}} h_{\epsilon, Y}(t) |E(z, 1/2 + it)|^2 dt \right) d\mu(z) + O(1).$$

To get better control of  $h_{\epsilon, Y}$  we compute  $h_R(t)$  explicitly in terms of special functions (see [3, (2.8), (2.9)] and subsequent discussion). We have

(7.13)

$$h_R(t) = \sqrt{2\pi \sinh R} \cdot \Re \left( e^{itR} \frac{\Gamma(it)}{\Gamma(3/2 + it)} F(-1/2, 3/2, 1 - it, (1 - e^{2R})^{-1}) \right),$$

where  $F$  is the Gauss hypergeometric function. For  $t$  real and nonzero

$$(7.14) \quad F(-1/2, 3/2, 1 - it, (1 - e^{2R})^{-1}) = 1 + O(\min(1, |t|^{-1})e^{-2R})$$

and Stirling's approximation [17, (5.113)] gives for  $|t| \geq 1$

$$\frac{\Gamma(it)}{\Gamma(3/2 + it)} = e^{-\frac{3\pi i}{4} \text{sgn} t} |t|^{-3/2} (1 + O(|t|^{-1})).$$

Inserting these expressions in (7.11) we find that

$$(7.15) \quad h_{\epsilon,Y}(t) = \Re \left( e^{i(tY - \frac{3\pi}{4} \text{sgnt})} \frac{\widehat{\psi}_\epsilon(t)}{|t|^{3/2}} \right) + O(|t|^{-5/2})$$

for  $|t| \geq 1$ . Since  $\widehat{\psi}_\epsilon(t) = O_m((\epsilon|t|)^{-m})$  for  $m \in \mathbb{N}$ , we have  $h_{\epsilon,Y}(t) = O_\epsilon(|t|^{-5/2})$ .

We recall that for  $\text{PSL}_2(\mathbb{Z})$  the scattering function  $\phi(s)$  has the special value  $\phi(1/2) = -1$ . This follows from the explicit calculation of  $\phi(s) = \xi(2-2s)/\xi(2s)$ , where  $\xi(s)$  is the completed Riemann zeta function. This implies, through the functional equation  $E(z, s) = \phi(s)E(z, 1-s)$  that the Eisenstein series vanishes identically at  $s = 1/2$ .

**Lemma 7.6.** *Let  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . For positive  $f$  as above*

$$\int_{\mathbb{R}} h_{\epsilon,Y}(t) \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z) dt = O_f(1).$$

uniformly in  $0 < \epsilon < 1$ .

*Proof.* We consider

$$\varphi(t) = \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 d\mu(z).$$

We have  $\varphi(0) = 0$  and using the Maaß–Selberg relations we find that  $\varphi(t) = O(\log(2 + |t|))$  for  $t \in \mathbb{R}$ . This is where we use crucially that  $\Gamma = \text{PSL}_2(\mathbb{Z})$ . We have

$$(7.16) \quad \int_{\mathbb{R}} h_{\epsilon,Y}(t) \int_{\Gamma \backslash \mathbb{H}} f(z) |E(z, 1/2 + it)|^2 dt = \int_{\mathbb{R}} h_{\epsilon,Y}(t) \varphi(t) dt.$$

To bound (7.16) we notice that by (7.13) and (7.14) that

$$\frac{h_R(t)}{\sqrt{2\pi \sinh(R)}} \varphi(t) = O_f \left( \left| \frac{\Gamma(it)}{\Gamma(3/2 + it)} \varphi(t) \right| \right).$$

Since  $|E(z, 1/2 + it)|^2 = E(z, 1/2 + it)E(z, 1/2 - it)$  is meromorphic as a function of  $t \in \mathbb{C}$ , and holomorphic for  $t \in \mathbb{R}$  and  $\varphi(0) = 0$ , the function

$$\frac{\Gamma(it)}{\Gamma(3/2 + it)} \varphi(t)$$

is holomorphic for  $t \in \mathbb{R}$ , i.e. the pole of the  $\Gamma$ -function in the numerator cancels with the zero of  $\varphi(t)$ . Moreover, we have the bound

$$\frac{\Gamma(it)}{\Gamma(3/2 + it)} \varphi(t) = O_f((1 + |t|)^{-3/2} \log(2 + |t|)).$$

It follows that

$$\int_{\mathbb{R}} h_{\epsilon,Y}(t) \varphi(t) dt = O_f \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \psi_\epsilon(R - Y) \left| \frac{\Gamma(it)}{\Gamma(3/2 + it)} \varphi(t) \right| dR dt \right) = O_f(1)$$

uniformly in  $\epsilon$ .  $\square$

**Lemma 7.7.** *Let  $\nu > 0$ . Then for every  $k$  and  $R > 0$  there exist  $\epsilon \in (0, 1)$ , and  $Y_0 > R$  such that*

$$-E_\epsilon(Y_0) > k(\log(Y_0))^{1/4-\nu}.$$

*Proof.* It follows from Lemma 7.6, (7.12) that

$$E_\epsilon(Y) = \sum_{1 \leq |t_j|} h_{\epsilon, Y}(t_j) \int_{\Gamma \backslash \mathbb{H}} f(z) |u_j(z)|^2 d\mu(z) + O_f(1),$$

where we have used that for  $\Gamma = \text{PSL}_2(\mathbb{Z})$  there are no non-zero eigenvalues with  $|t_j| < 1$ . Using (7.15),  $\int_{\Gamma \backslash \mathbb{H}} f |u_j|^2 d\mu \leq \|f\|_\infty$ , and Weyl's law this equals

$$(7.17) \quad \sum_{1 \leq |t_j|} \Re \left( e^{i(t_j Y - \frac{3\pi}{4} \text{sgn } t_j)} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \right) \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) + O_f(1).$$

We split the sum at  $T$ , and bound the partial sum and the tail separately. The precise value of  $T$  will be chosen later. We use  $\int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) \leq \|f\|_\infty$  and  $\widehat{\psi}_\epsilon(t) = O_m((\epsilon |t|)^{-m})$  for any  $m$  to bound the tail as follows:

$$\begin{aligned} \sum_{|t_j| > T} \Re \left( e^{i(t_j Y - \frac{3\pi}{4} \text{sgn } t_j)} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \right) \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) &= O_{f,m}(\epsilon^{-m} \sum_{|t_j| > T} |t_j|^{-(3/2+m)}) \\ &= O_{f,m}(\epsilon^{-m} T^{1/2-m}), \end{aligned}$$

where we have used Weyl's law to estimate the sum of the series.

To analyze the partial sum Lemma 7.2 allows us to choose  $Y = Y_0 > 0$  depending on  $T, R, V$  with  $R \leq Y_0 \leq RV^{N(T)-1}$  such that  $|e^{it_j Y_0} - 1| < V^{-1}$  for all  $1 < |t_j| \leq T$ . Here  $N(T) - 1$  is the number of such  $t_j$ 's. Without loss of generality we can assume that  $t_j > 0$ . Using the addition formula for cosine we see that

$$\left| \cos(Y_0 t_j - 3\pi/4) - (-\sqrt{2}/2) \right| \leq V^{-1}.$$

Using Weyl's law we deduce that

$$\begin{aligned} \sum_{1 \leq t_j \leq T} \Re \left( e^{i(t_j Y_0 - \frac{3\pi}{4} \text{sgn } t_j)} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \right) \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) \\ = -\frac{\sqrt{2}}{2} \sum_{1 \leq t_j \leq T} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) + O_f(T^{1/2}/V). \end{aligned}$$

Since  $\widehat{\psi}(0) = 1$  we have, by an appropriate choice of  $0 < \tau < 1$ , that  $\widehat{\psi}(t) \geq 1/2$  for  $|t| \leq \tau$ . It follows that when  $|t| \leq \tau/\epsilon$  we have  $\widehat{\psi}_\epsilon(t) \geq 1/2$ . We note that all terms in the sum above are non-negative. Therefore, for

$\tau/\epsilon < T$ , we have

$$\begin{aligned} \frac{-\sqrt{2}}{2} \sum_{1 \leq t_j \leq T} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) &\leq \frac{-\sqrt{2}}{2} \sum_{1 \leq t_j \leq \tau/\epsilon} \frac{\widehat{\psi}_\epsilon(t_j)}{|t_j|^{3/2}} \int_{\Gamma \backslash \mathbb{H}} f d\mu_j(z) \\ &\leq -C\epsilon^{-1/2}. \end{aligned}$$

Here we have used Lemma 7.3, and  $C$  is some strictly positive constant depending only on  $f$ ,  $\tau$ ,  $m$  and  $\Gamma$ . We note that in Lemma 7.3 the contribution from the continuous spectrum is  $O(T \log T)$ , as follows from (6.2).

To summarize we have proved that there exist  $Y_0$ ,  $R \leq Y_0 \leq RV^{N(T)-1}$  with

$$(7.18) \quad -E_\epsilon(Y_0) \geq C\epsilon^{-1/2} + O_{m,f}((1 + \epsilon^{-m}T^{1/2-m} + T^{1/2}/V).$$

Let  $\epsilon, T$  be chosen such that  $T^{1/2}/V = \epsilon^{-m}T^{1/2-m} = 1$ . If we choose  $V$  such that  $V > R$  (note that this puts an upper bound on  $\epsilon$ ) and if we assume that  $R$  from the beginning was sufficiently large we have

$$(7.19) \quad -E_\epsilon(Y_0) \geq \frac{C}{2}V^{1-1/(2m)}.$$

Since  $V > R$  we have  $Y_0 \leq V^{N(T)} \leq V^{c_1 T^2} = V^{c_1 V^4}$  which forces  $c_2 V^4 \geq \log(Y_0)/\log \log(Y_0)$ . Now we choose  $m$  to satisfy  $(8m)^{-1} < \nu/2$ , so that

$$V^{1-1/(2m)} \geq (c_2^{-1} \log(Y_0)/\log \log(Y_0))^{\frac{1}{4}(1-1/(2m))} \geq c_3 \log(Y_0)^{\nu/2} (\log Y_0)^{1/4-\nu}.$$

The proof is complete once we observe that we could assume that  $R$  had been chosen such that  $c_3 \log(Y_0)^{\nu/2} > k$  for  $Y_0 \geq R$ .  $\square$

*Proof of theorem 7.1.* Proof by contradiction. Assume that for some  $\nu > 0$

$$\frac{N_f(X) - \pi \bar{f} X}{X^{1/2}} = O((\log \log X)^{1/4-\nu}).$$

It follows from (7.9) and  $X = 2 \cosh Y$  that

$$E(Y) = O((\log Y)^{1/4-\nu}),$$

and, therefore,

$$|E_\epsilon(Y)| \leq \int_{\mathbb{R}} \psi_\epsilon(R - Y) |E(R)| dR \leq K(\log(Y + \epsilon))^{1/4-\nu} = O((\log Y)^{1/4-\nu})$$

uniformly for all  $\epsilon$ . But this contradicts Lemma 7.7.  $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET,  
LONDON WC1E 6BT, UNITED KINGDOM

*E-mail address:* `i.petridis@ucl.ac.uk`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVER-  
SITETSPARKEN 5, 2100 COPENHAGEN Ø, DENMARK

*E-mail address:* `risager@math.ku.dk`

# APPENDIX: NUMERICAL INVESTIGATION OF EXPONENTIAL SUMS OVER EIGENVALUES IN $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$

NIKO LAAKSONEN

Let  $\lambda_j = \frac{1}{4} + t_j^2$  be the eigenvalues of  $\Delta$  in  $\Gamma \backslash \mathbb{H}$ , where  $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ . For  $X > 1$ , we define the following sum

$$S(T, X) = \sum_{|t_j| \leq T} X^{it_j},$$

which is symmetrised by including both  $t_j$  and  $-t_j$ . Petridis and Risager [7, Conjecture 2.2] conjecture that up to a factor of the order of  $X^\epsilon$ , the sum has square root cancellation in  $T$ .

**Conjecture.** *For every  $\epsilon > 0$  and  $X > 1$  we have*

$$S(T, X) \ll_\epsilon T^{1+\epsilon} X^\epsilon.$$

We report on the numerical investigation of the function  $S(T, X)$  and prove a theorem about its behaviour as  $T \rightarrow \infty$  and  $X > 1$  is fixed. Our investigation resulted in the following observations.

**Experimental Observation 1.** *The growth of  $S(T, X)$  is consistent with the conjecture.*

**Experimental Observation 2.** *For a fixed  $X > 1$ ,  $S(T, X)$  has a peak of order  $T$  whenever  $X$  is equal to a power of a norm of a primitive hyperbolic class of  $\Gamma$  or an even power of a prime number  $p \in \mathbb{N}$ .*

Experimental Observation 2 is also in agreement with the results of Chazaraïn [2] that for the wave kernel the singularities occur at the lengths of closed geodesics (or in our case when  $\log X$  is a multiple of a length of a prime geodesic). The peaks at even powers of rational primes are due to the scattering determinant  $\varphi$ . Experimental Observation 2 leads us to prove asymptotics for  $S(T, X)$  for a fixed  $X > 1$ . Let  $\Lambda(X)$  be the von Mangoldt function extended to  $\mathbb{R}$  by defining it to be 0 when  $X$  is not equal to a power of a prime number. We also define a similar function  $\Lambda_\Gamma$  for the norms of hyperbolic classes of  $\mathrm{PSL}_2(\mathbb{Z})$  given by

$$\Lambda_\Gamma(X) = \begin{cases} \log(N(\mathfrak{p})), & \text{if } X = N(\mathfrak{p})^\ell, \ell \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $|F|$  be the volume of the fundamental domain of  $\Gamma \backslash \mathbb{H}$ . We prove the following theorem.

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**Theorem 1.** *For a fixed  $X > 1$ , we have*

$$S(T, X) = \frac{|F|}{\pi} \frac{\sin(T \log X)}{\log X} T + \frac{T}{\pi} (X^{1/2} - X^{-1/2})^{-1} \Lambda_{\Gamma}(X) \\ + \frac{2T}{\pi} X^{-1/2} \Lambda(X^{1/2}) + O(T/\log T),$$

as  $T \rightarrow \infty$ .

*Proof.* Let  $\psi$  be a positive even test function supported on  $[-1, 1]$  with  $\int \psi = 1$ . Then define  $\psi_{\epsilon}(x) = \epsilon^{-1} \psi(x/\epsilon)$ . So  $\psi_{\epsilon}$  is supported on  $[-\epsilon, \epsilon]$  and  $\int \psi_{\epsilon} = 1$ . Also, let  $G$  be the convolution  $G(r) = (\mathbb{1}_{[-T, T]} * \psi_{\epsilon})(r)$  for some  $\epsilon > 0$  to be chosen later. Define a function  $h$ , depending on  $T$ ,  $X$  and  $\epsilon$ , given by  $h(r) = G(r)(X^{ir} + X^{-ir})$ . Let  $g$  be the Fourier transform of  $h$  and denote the determinant of the scattering matrix  $\Phi$  by  $\varphi$ . Then the Selberg Trace Formula [3, Theorem 10.2] says that

$$\mathcal{S}(T, X) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{-\varphi'}{\varphi} \left( \frac{1}{2} + ir \right) dr = I(T, X) + H(T, X) + E(T, X) + L(T, X),$$

where

$$\begin{aligned} \mathcal{S}(T, X) &= \sum_{t_j > 0} h(t_j), \\ I(T, X) &= \frac{|F|}{4\pi} \int_{-\infty}^{\infty} h(r) r \tanh(\pi r) dr, \\ H(T, X) &= \sum_{\mathfrak{p}} \sum_{\ell=1}^{\infty} \left( N(\mathfrak{p})^{\ell/2} - N(\mathfrak{p})^{-\ell/2} \right)^{-1} g(\ell \log N(\mathfrak{p})) \log N(\mathfrak{p}), \\ E(T, X) &= \sum_{\mathcal{R}} \sum_{0 < \ell < m} \left( 2m \sin \frac{\pi \ell}{m} \right)^{-1} \int_{-\infty}^{\infty} h(r) \frac{\cosh \pi \left( 1 - \frac{2\ell}{m} \right) r}{\cosh \pi r} dr, \\ L(T, X) &= \frac{h(0)}{4} \operatorname{Tr}(I - \Phi(\tfrac{1}{2})) - h_{\Gamma} g(0) \log 2 - \frac{h_{\Gamma}}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'(1+ir)}{\Gamma(1+ir)} dr, \end{aligned}$$

where  $\mathfrak{p}$  and  $\mathcal{R}$  range over the primitive hyperbolic and elliptic classes of  $\operatorname{PSL}_2(\mathbb{Z})$ , respectively, and  $h_{\Gamma}$  is the number of cusps of  $\Gamma$ . First observe that  $S(T, X) = \mathcal{S}(T, X) + O(T\epsilon)$ , so we can work with  $\mathcal{S}$ . For the identity motion we have

$$\begin{aligned} I(T, X) &= \frac{|F|}{2\pi} \int_{-\infty}^{\infty} G(r) \cos(r \log X) r \tanh \pi r dr \\ &= \frac{|F|}{\pi} \int_0^{\infty} G(r) \cos(r \log X) r \left( 1 - \frac{2}{e^{2\pi r} + 1} \right) dr \\ &= \frac{|F|}{\pi} (I_1(T, X) + I_2(T, X)). \end{aligned}$$

From  $I_1$  we obtain a part of the main term:

$$\begin{aligned} I_1(T, X) &= \int_0^\infty G(r) \cos(r \log X) r \, dr \\ &= \left( \int_0^{T-\epsilon} + \int_{T-\epsilon}^{T+\epsilon} \right) G(r) \cos(r \log X) r \, dr \\ &= I_{11} + I_{12}, \end{aligned}$$

since  $G$  is even and supported on  $[-T - \epsilon, T + \epsilon]$ . Then

$$\begin{aligned} I_{11} &= \int_0^{T-\epsilon} \cos(r \log X) r \, dr = \frac{\sin((T - \epsilon) \log X)}{\log X} (T - \epsilon) + O(1), \\ I_{12} &\ll \int_{T-\epsilon}^{T+\epsilon} r \, dr = O(T\epsilon). \end{aligned}$$

Also,

$$I_2(T, X) = - \int_0^\infty G(r) \cos(r \log X) r \frac{2}{e^{2\pi r} + 1} \, dr \ll \int_0^\infty r e^{-2\pi r} \, dr = O(1).$$

For  $g(r)$  we compute

$$\begin{aligned} g(r) &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-irt} h(t) \, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty G(t) e^{-irt} (e^{it \log X} + e^{-it \log X}) \, dt \\ &= \frac{1}{2\pi} \left( \widehat{G} \left( \frac{r - \log X}{2\pi} \right) + \widehat{G} \left( \frac{r + \log X}{2\pi} \right) \right). \end{aligned}$$

So in particular  $g(\ell \log N(\mathfrak{p})) \sim T/\pi$  if  $X = N(\mathfrak{p})^\ell$  and decays as  $O((\ell \log N(\mathfrak{p}))^{-k-1} \epsilon^{-k})$  otherwise, for any  $k \in \mathbb{N}$ . For the elliptic terms we need to evaluate

$$\int_{-\infty}^\infty h(r) \frac{\cosh \pi(1 - \frac{2\ell}{m})r}{\cosh \pi r} \, dr \ll \int_0^\infty \frac{e^{-2\pi r \ell/m} + e^{-2\pi r}}{1 + e^{-2\pi r}} \, dr = O(1).$$

Hence  $E(T, X)$  is bounded. By the explicit formula of  $\varphi'/\varphi$  for  $\mathrm{PSL}_2(\mathbb{Z})$ , [3, 3.24], we have

$$\begin{aligned} \int_{-\infty}^\infty h(r) \frac{-\varphi'}{\varphi} \left( \frac{1}{2} + ir \right) \, dr &= \int_{-\infty}^\infty h(r) \left( -2 \log \pi + \frac{\Gamma'(\frac{1}{2} \pm ir)}{\Gamma(\frac{1}{2} \pm ir)} + 2 \frac{\zeta'(1 \pm 2ir)}{\zeta(1 \pm 2ir)} \right) \, dr. \\ &= C_1 + C_2 + C_3. \end{aligned}$$

The integral  $C_1$  is the Fourier transform of  $G$  and is thus bounded. For  $C_2$  we use Stirling asymptotics to get

$$C_2 = \int_{-\infty}^\infty h(r) \log \left( \frac{1}{4} + r^2 \right) \, dr + O(1).$$

This is  $O(\log T)$ . The same computation shows that  $L(T, X) = O(\log T)$ . The remaining part of the main term comes from  $C_3$ . We first expand  $h$  and isolate the important terms:

$$C_3 = 2 \left( \int_{-T-\epsilon}^{-T+\epsilon} + \int_{-T+\epsilon}^{T-\epsilon} + \int_{T-\epsilon}^{T+\epsilon} \right) (X^{ir} + X^{-ir}) G(r) \frac{\zeta'(1 \pm 2ir)}{\zeta(1 \pm 2ir)} \, dr.$$

The first and third integrals are bounded by  $O(\epsilon \log T)$ . Notice that  $G(r) = 1$  in the range of the second integral, hence we can write it as

$$2 \int_{\frac{1}{2}-(T-\epsilon)i}^{\frac{1}{2}+(T-\epsilon)i} (X^{s-1/2} + X^{1/2-s}) \left( \frac{\zeta'(2s)}{\zeta(2s)} + \frac{\zeta'(2-2s)}{\zeta(2-2s)} \right) ds.$$

We separate this into two integrals by adding and subtracting the singular part:

$$\begin{aligned} C_3 &= 2 \int_{\frac{1}{2}-(T-\epsilon)i}^{\frac{1}{2}+(T-\epsilon)i} (X^{s-1/2} + X^{1/2-s}) \left( \frac{\zeta'(2s)}{\zeta(2s)} - \frac{1}{2s-1} \right) ds \\ &\quad + 2 \int_{\frac{1}{2}-(T-\epsilon)i}^{\frac{1}{2}+(T-\epsilon)i} (X^{s-1/2} + X^{1/2-s}) \left( \frac{\zeta'(2-2s)}{\zeta(2-2s)} - \frac{1}{(2-2s)-1} \right) ds \\ &= 2(C_{31} + C_{32}). \end{aligned}$$

For the first integral we move the contour to  $\Re s = 1$  and for the second one to  $\Re s = 0$ . It is easy to see that the top and bottom parts of the contours yield  $O(\log T)$ . For the line at  $\Re s = 1$  we get

$$\begin{aligned} C_{31} &= \int_{1-(T-\epsilon)i}^{1+(T-\epsilon)i} (X^{s-1/2} + X^{1/2-s}) \left( \frac{\zeta'(2s)}{\zeta(2s)} - \frac{1}{2s-1} \right) ds \\ &= \int_{-T+\epsilon}^{T-\epsilon} (X^{1/2+ir} + X^{-1/2-ir}) \left( \frac{\zeta'(2+2ir)}{\zeta(2+2ir)} - \frac{1}{1+2ir} \right) dr, \end{aligned}$$

For the rest of the proof we will follow an argument similar to [5, Hilfssatz 2]. We start by writing out the Dirichlet series:

$$\begin{aligned} C_{31} &= - \int_{-T+\epsilon}^{T-\epsilon} X^{1/2+ir} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{2+2ir}} dr + O(\log T) \\ &= - \sum_{n \neq \sqrt{X}} \frac{\sqrt{X} \Lambda(n)}{n^2} \int_{-T+\epsilon}^{T-\epsilon} \left( \frac{X}{n^2} \right)^{ir} dr - X^{-1/2} \Lambda(X^{1/2}) \int_{-T+\epsilon}^{T-\epsilon} dr + O(\log T). \end{aligned}$$

Since  $X > 1$ , the term in  $C_{31}$  with the negative exponent gets absorbed into the error term. Hence,

$$\begin{aligned} &\left| \int_{-T+\epsilon}^{T-\epsilon} X^{1/2+ir} \frac{\zeta'(2+2ir)}{\zeta(2+2ir)} dr + 2X^{-1/2} \Lambda(X^{1/2})(T-\epsilon) \right| \\ &\leq \sum_{n \neq \sqrt{X}} \frac{\sqrt{X} \Lambda(n)}{n^2} \left| \frac{\left( \frac{X}{n^2} \right)^{i(T-\epsilon)} - \left( \frac{X}{n^2} \right)^{-i(T-\epsilon)}}{\log \frac{X}{n^2}} \right| \\ &\ll 2\sqrt{X} \left| \frac{\zeta'(2)}{\zeta(2)} \right|. \end{aligned}$$

So we see that  $C_{31} = -2X^{-1/2} \Lambda(X^{1/2})(T-\epsilon) + O(\log T)$ . A similar argument shows that  $C_{32}$  has the same asymptotics. Letting  $\epsilon = 1/\log T$  concludes the proof.  $\square$

We will now present plots of  $S(T, X)$  in terms of both  $T$  and  $X$ . In Figures 1 to 3 we have fixed  $T = 800$  with  $X \rightarrow \infty$ , while in Figure 4 we are

fixing  $X$  with  $T \rightarrow \infty$ . Taking into account the conjecture, we plot the normalised sum  $\Sigma(T, X) = S(T, X)T^{-1}$ . In Figure 4 we present a comparison for different powers of  $T$ , which suggests that  $1 + \epsilon$  is the correct exponent. The programs used for the plots are available on the website [4]. We used 53 000 eigenvalues from the data of Then [8] with 13 decimal digit precision. We have also used the data of Booker and Strömbergsson related to [1], which has a much higher precision of 53 decimal digits for 2 280 eigenvalues. We verify that the computations are robust, that is, the number of eigenvalues or their precision has no significant impact on our calculations. More details are available on the website [4].

Recall that we expect a peak of order  $T$  at all even prime powers as well as powers of the norms of the primitive hyperbolic classes. The first few norms (up to 8 decimals) are given by

$$\begin{array}{ll} g_1 = 6.85410196 & g_5 = 46.97871376 \\ g_2 = 13.92820323 & g_6 = 61.98386677 \\ g_3 = 22.95643924 & g_7 = 78.98733975 \\ g_4 = 33.97056274 & g_8 = 97.98979486. \end{array}$$

These can be computed by expressing the norm in terms of the trace (see e.g. [3, pg. 146]).

We start by considering  $\Sigma(T, X)$  in terms of  $X$ .

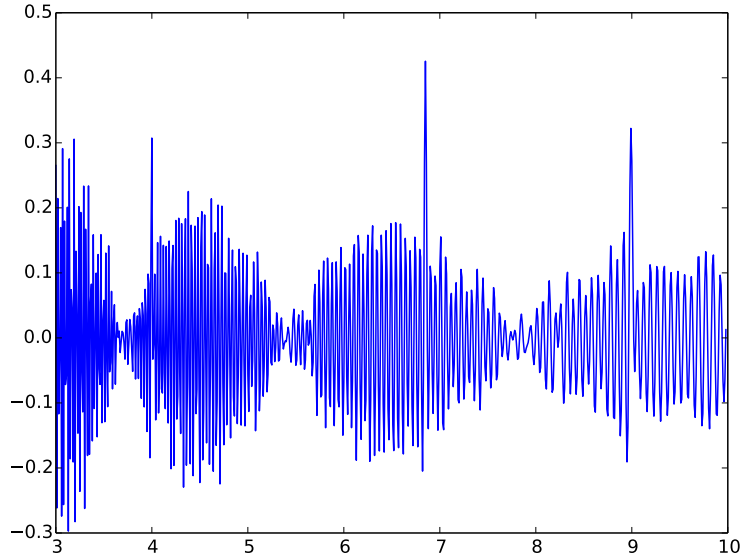


FIGURE 1.  $\Sigma(T, X)$  in terms of  $X$  for  $X \in [3, 10]$ .

This clearly shows peak points at  $X = 4 = 2^2$ ,  $X = g_1$  and  $X = 9 = 3^2$ . In the following plot we can see the peak points  $X = g_2$  and  $X = 16 = 2^4$ :

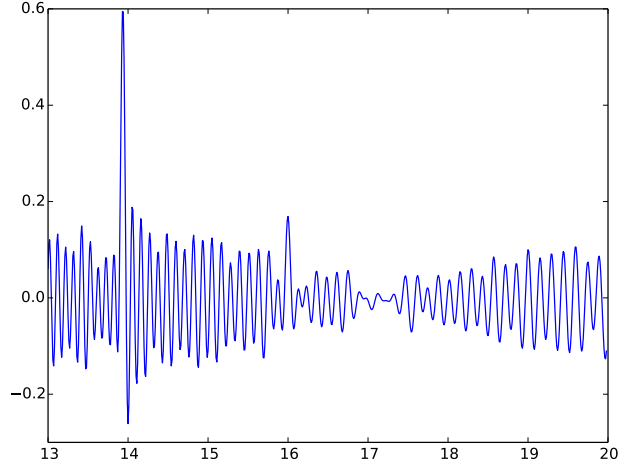


FIGURE 2.  $\Sigma(T, X)$  in terms of  $X$  for  $X \in [13, 20]$ .

Figures 1 and 2 verify Theorem 1 numerically in accordance with Experimental Observation 2. In Figure 3 we look at  $\Sigma(T, X)$  for  $X$  in a much larger interval. The graph agrees with Experimental Observation 1. On the other hand we cannot dispose of  $X^\epsilon$  in the conjecture. The frequencies  $t_j$  are conjecturally linearly independent over  $\mathbb{Q}$ , which makes  $S(T, X)$  the partial sums of an almost periodic function. Therefore, for a choice of arbitrarily large  $X$ , compared to  $T$ ,  $S(T, X)$  will be of size  $T^2$ .

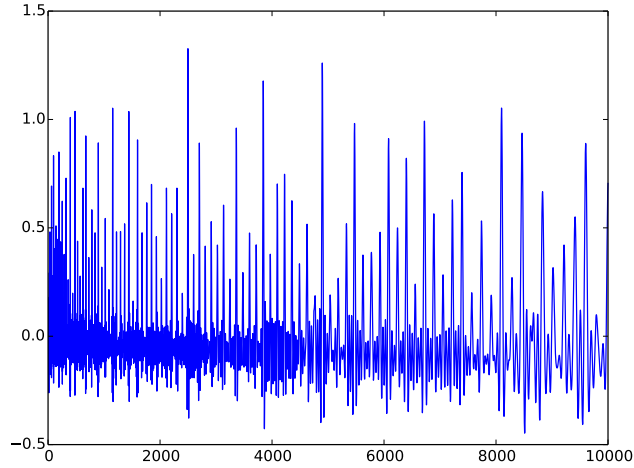


FIGURE 3.  $\Sigma(T, X)$  in terms of  $X$  for  $X \in [100, 10\,000]$ .

In Theorem 1 the asymptotics show an oscillatory term with an amplitude of order  $T$  coming from the identity motion. We subtract it from  $S(T, X)$

and define

$$\tilde{\Sigma}(T, X) = S(T, X) - \frac{|F|}{\pi} \frac{\sin(T \log X)}{\log X} T.$$

We plot  $\tilde{\Sigma}(T, X)$  in terms of  $T$  at  $X = 49$ , which is one of the peak points.

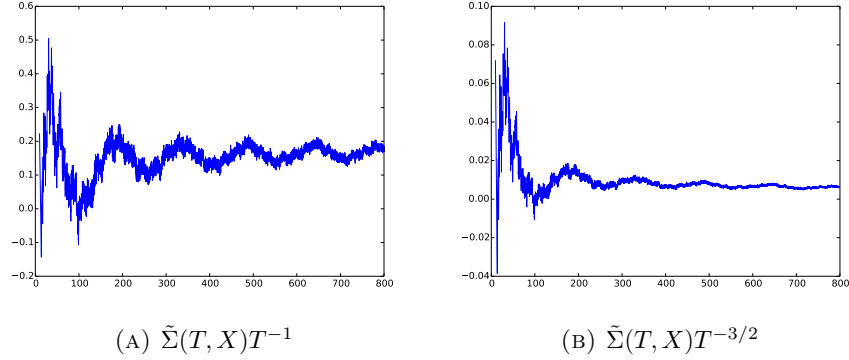


FIGURE 4. Different normalisations of  $\tilde{\Sigma}(T, X)$  at  $X = 49$ .

Notice that clearly the normalisation  $T^{-1}$  seems to be closer to the correct one, which is evidence towards our Experimental Observation 1.

It is of interest to compare the behaviour of  $S(T, X)$  with a similar sum over the Riemann zeros. Landau [5, Satz 1] showed that for a fixed  $x > 1$ , if  $\rho = \beta + i\gamma$  is a non-trivial zero of  $\zeta(s)$ , we have the formula

$$\sum_{0 < \gamma < T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O(\log T). \quad (1)$$

We call the left-hand side of (1)  $Z(T, X)$ . We used our program with 10 000 zeros of  $\zeta(s)$  to 9 decimal places, provided by Odlyzko [6]. With our program we obtain the following plot for the normalized sum  $T^{-1}Z(T, X)$ . Here blue denotes the real part and green the imaginary part of the sum.

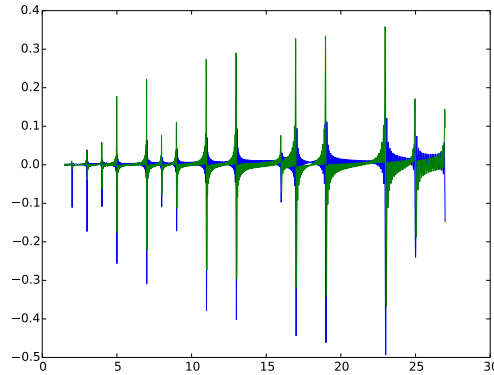


FIGURE 5.  $x \in [1.5, 30]$

Notice that  $S(T, X)$  is the analogue of the real part of  $Z(T, X)$  only. Since the Selberg Trace Formula demands that the test function is even, we cannot analyse the imaginary part directly. For numerical study of this we again refer the reader to the website [4].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET,  
LONDON WC1E 6BT, UNITED KINGDOM  
*E-mail address:* `n.laaksonen@ucl.ac.uk`